Diversification, Volatility, and Surprising Alpha

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- Suppose we wish to calculate the average annual return of an investment over several years, where the annual returns are given by $r_1, r_2, \ldots, r_n$.
- Several common methods are available.
  1. *Arithmetic return:* $\frac{1}{n}\Big((1 + r_1) + \cdots + (1 + r_n)\Big) - 1.$
  2. *Geometric return:* $\sqrt[n]{(1 + r_1) \times \cdots \times (1 + r_n)} - 1.$
  3. *Logarithmic return:* $\frac{1}{n}\Big(\log(1 + r_1) + \cdots + \log(1 + r_n)\Big).$

Poll A

(How) are the different returns ordered?

1. Arithmetic return always greater than geometric and logarithmic returns; not more can be said.
2. Arithmetic return always greater than geometric return, which is always larger than logarithmic return.
3. The ordering depends on the series $(r_1, r_2, \ldots)$.

Different averages

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1.$
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Jensen's inequality yields

$$\text{arithmetic return} \;\geq\; \text{geometric return} \;\geq\; \text{logarithmic return}.$$

The dynamics of return

Let $S(t)$ represent the price of a stock at time t. Assume that

$$\mathrm{d}S(t) = S(t)\Big[b\mathrm{d}t + \sigma\mathrm{d}W(t)\Big].$$

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- Itô's formula implies that

$$\mathrm{d}\log S(t) = g\,\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where $g = b - \frac{1}{2}\sigma^2$ is the *rate of log-return*, or *growth rate*, of S. This '*volatility drag*' in words:

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- The process g determines the long-term behavior of S:

  $$\lim_{T\uparrow\infty} \frac{1}{T}\log S(T) = g.$$

Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio π with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time t. Then the portfolio return satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{d} \pi_i(t) \, \frac{\mathrm{d}S_i(t)}{S_i(t)}$$

(Markowitz (1952)).

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$$\mathrm{d} \log V^\pi(t) = \sum_{i=1}^{d} \pi_i(t) \, \mathrm{d} \log S_i(t) + \gamma_\pi^*(t) \, \mathrm{d}t,$$

where $\gamma_\pi^* \geq 0$ if the portfolio is long-only. (Fernholz and Shay (1982)).

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γ_π^* **depends only on the covariance structure of** S**.**

Details: the dynamics of portfolio log-return

$$
\begin{aligned}
\mathrm{d}\log V^\pi(t) &= \frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} - \frac{1}{2}\sigma_\pi^2(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{d} \pi_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - \frac{1}{2}\sigma_\pi^2(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{d} \pi_i(t)\Big(\mathrm{d}\log S_i(t) + \frac{1}{2}\sigma_i^2(t)\,\mathrm{d}t\Big) - \frac{1}{2}\sigma_\pi^2(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{d} \pi_i(t)\mathrm{d}\log S_i(t) + \gamma_\pi^*(t)\,\mathrm{d}t,
\end{aligned}
$$

with $$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^{d}\pi_i(t)\sigma_i^2(t) - \sigma_\pi^2(t)\Big).$$

Decomposition of portfolio log-return

There is a natural decomposition of the log-return of a portfolio into two components. For the interval $[0, T]$,

$$\begin{aligned}\text{Log-return} &= \int_0^T \sum_{i=1}^d \pi_i(t)\,\mathrm{d}\log S_i(t) + \int_0^T \gamma_\pi^*(t)\mathrm{d}t \\ &=: A_\pi(T) + \Gamma_\pi(T)\end{aligned}$$

In words:

$$\begin{aligned}\text{Log-return} &= \text{weighted average stock log-return} \\ &\quad + \text{excess growth rate;}\end{aligned}$$

$$\text{EGR} = \frac{\text{weighted average stock variance} - \text{portfolio variance}}{2}.$$

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An important empirical property of equity markets

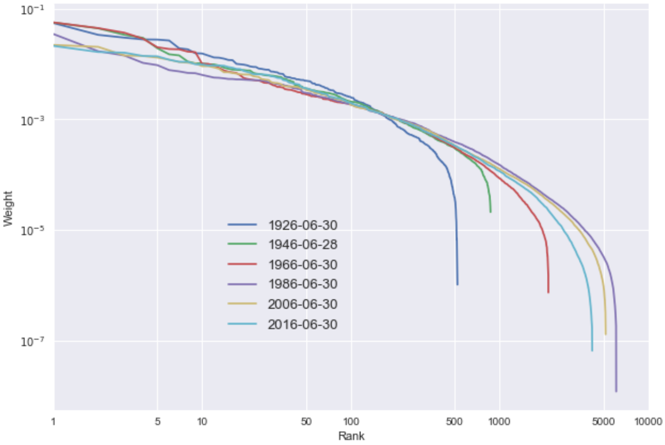


Figure: The capital distribution curve — Market weights against ranks on logarithmic scale, 1926–2016.

Poll B

How many companies were at some point the largest (in the US market) since 1926? (Currently Apple is the largest.)

1. 6
2. 11
3. 19
4. 32

Answer: 11

Largest at some point:

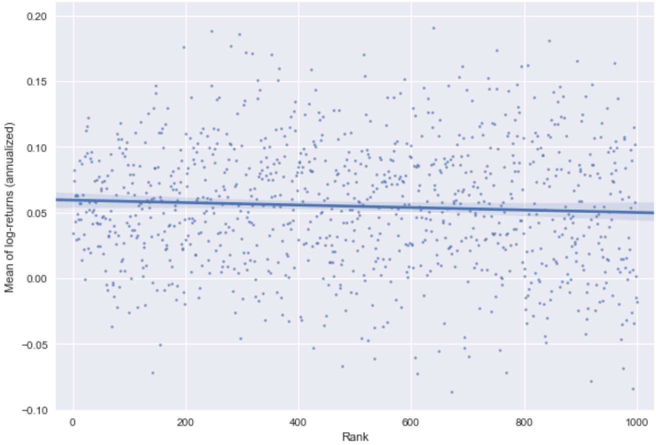
- Amazon
- Apple
- ATT
- DuPont
- Exxon
- GE
- GM
- IBM
- Microsoft
- Philip Morris
- Walmart

Rank-based analysis of logarithmic returns

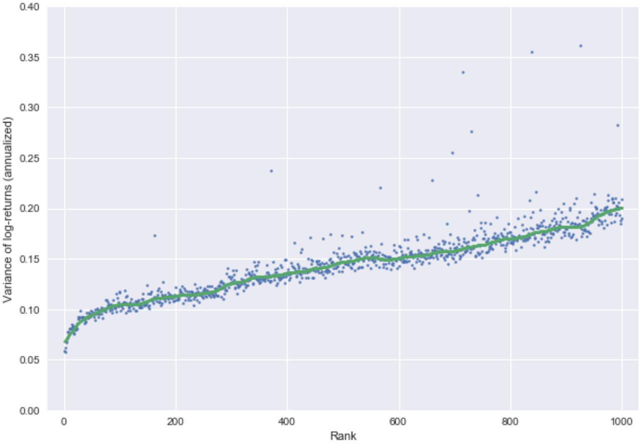
- Let $r_t(i)$ be the rank of $S_i(t)$.
- Define the *average rank-based growth rates* $\boldsymbol{g}_k$ over $[0, T]$ by

$$\boldsymbol{g}_k = \frac{1}{T} \int_0^T \sum \mathbf{1}_{\{r_t(i)=k\}} \mathrm{d} \log S_i(t).$$

Estimated g_k, 1962–2016



Sample variance of logarithmic returns



Rank-based analysis of logarithmic returns

In a stable system:

$$\mathsf{E}\left[\mathrm{d}\log X_i(t)\middle| r_t(i) = k\right] = \boldsymbol{g}_k \mathrm{d}t \approx \boldsymbol{g}\mathrm{d}t.$$

Hence,

$$\begin{aligned}
\mathsf{E}\left[\mathsf{A}_\pi(T)\right] &= \mathsf{E}\left[\int_0^T \sum_{i=1}^d \pi_i(t)\mathrm{d}\log X_i(t)\right] \\
&\simeq \int_0^T \sum_{i=1}^d \mathsf{E}\left[\pi_i(t)\right]\mathsf{E}\left[\mathrm{d}\log X_i(t)\right] \\
&\simeq \int_0^T \sum_{i=1}^d \mathsf{E}\left[\pi_i(t)\right]\boldsymbol{g}\mathrm{d}t \\
&= T\boldsymbol{g}.
\end{aligned}$$

A paper on "surprising alpha"

Arnott et al. (2013) test several naïve, non-optimized portfolio strategies versus a capitalization-weighted benchmark of the largest 1000 U.S. stocks over the period from 1964 to 2012.

- All tested strategies have a higher return than the benchmark, and most have a higher Sharpe ratio.
- Capitalization-weighted portfolios are not well diversified.
- All tested strategies have more diversification into the smaller stocks than the capitalization-weighted index.

Our experimental setup

- We run an experiment on the largest 1000 U.S. stocks,
- using overlapping one-year periods starting each month from 1964-2012 (similar in spirit to Arnott (2013)).
- At the beginning of each month we choose the largest 1000 U.S. stocks and use their one-year returns over the following year to compute the strategy returns.
- Altogether there are 5384 different stocks which were, at the beginning of some month during this 49-year period, among the top 1000 stocks by market capitalization in the U.S.

5 representative naïve strategies

We implement the following strategies.

1. **Capitalization-weighted (CW):** stock weights proportional to their market capitalization.
2. **Equal-weighted (EW):** weight of each stock $= 1/1000$.
3. **Large-overweighted (LO):** stock weights proportional to the square of their market capitalization.
4. **Random-weighted (RW):** weights proportional to $[0, 1]$–uniformly distributed random variables.
5. **Inverse-random-weighted (IRW):** weights proportional to the reciprocals of $[0, 1]$–uniformly distributed random variables.

Poll C

Among these five strategies, which strategies perform best between 1964 and 2012 in the above setup (in particular, without transaction costs). Say in terms of the largest Sharpe ratio?

1. Random-weighted and Equal-weighted
2. Large-overweighted
3. Random-weighted and Inverse-random-weighted
4. Capitalization-weighted

The results

	CW(%)	**EW**(%)	**LO**(%)	**RW**(%)	**IRW**(%)
Log-return	9.12	10.98	7.46	10.98	10.46
v. CW		1.86	-1.66	1.86	1.34
$A_\pi(T)$	5.57	5.64	5.36	5.65	5.67
v. CW		.07	-.21	.08	.10
$\boldsymbol{\Gamma_\pi(\mathbf{T})}$	**3.87**	**5.82**	**2.19**	**5.82**	**5.18**
v. CW		1.95	-1.68	1.95	1.31
Arithmetic	10.97	13.33	9.15	13.33	13.34
v. CW		2.36	-1.82	2.36	2.37
σ_π	17.07	19.14	16.90	19.07	22.35
S.R.	.29	.38	.18	.38	.32

Table 1. CW = cap weight, EW = equal weight, LO = large-overweighted, RW = random weight, IRW = inverse random weight, S.R. = Sharpe ratio.

Conclusions

- The logarithmic return of a portfolio can be decomposed into two elements:
  - weighted average of the logarithmic returns of the stocks;
  - excess growth: depends only on the variances and covariances of the constituents, and is larger for more diversified portfolios.
- Most of the differences in the strategies' returns can be explained by differences in the excess growth component.

Many thanks!